

Homework #10

① Check if the following are linear transformations. If so, prove it. If not give an explicit example why not.

(a) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $T(x, y) = (x + 2y, 3x - y)$

(b) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $T(x, y, z) = (2x - y + z, y - 4z)$

(c) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T(x, y) = \sqrt{x^2 + y^2}$
This is the same as $T(\vec{v}) = \|\vec{v}\|$

(d) $T: M_{2,2} \rightarrow \mathbb{R}$ given by
 $T\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = 3a - 4b + c - d$

(e) $T: M_{2,2} \rightarrow \mathbb{R}$ given by
 $T\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = a^2 + b^2$

(f) $T: P_2 \rightarrow P_2$ given by
 $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(1+x) + a_2(1+x)^2$

(g) $T: P_2 \rightarrow P_2$ given by
 $T(a_0 + a_1x + a_2x^2) = (1+a_0) + (1+a_1)x + (1+a_2)x^2$

② Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation. Suppose we know that $T(1, 0, 0) = (2, 1, -1)$, $T(0, 1, 0) = (0, \pi, 2/3)$, and $T(0, 0, 1) = (-1, 0, 0)$. Find a formula for $T(x, y, z)$ in general.

③ Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by
 $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ -2x+4y \end{pmatrix}$. T is a linear transformation
 (you can verify that for extra practice).
 Let $\beta = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$. Then β is an ^{ordered} basis
 for \mathbb{R}^2 (you can verify for extra practice).

(a) Find $[T]_{\beta}$

(b) Verify that $[T]_{\beta} [\vec{x}]_{\beta} = [T(\vec{x})]_{\beta}$
 for $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

④ Let $T: P_2 \rightarrow P_1$ be the linear
 transformation defined by

$$T(a_0 + a_1x + a_2x^2) = (a_0 + a_1) - (2a_1 - 3a_2)x$$

Let $\beta = [1, x, x^2]$ and $\beta' = [1, x]$ be ordered
 bases for P_2 and P_1 , respectively.

(a) Calculate $[T]_{\beta}^{\beta'}$

(b) Show that $[T(\vec{v})]_{\beta'} = [T]_{\beta}^{\beta'} [\vec{v}]_{\beta}$

for $\vec{v} = 1 + x + x^2$ and $\vec{v} = 2x^2$.

⑤ Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operator defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$$

and let ~~$\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$~~ $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]$ be an ordered basis for \mathbb{R}^2 ,

(a) Calculate $[T]_{\beta}$

(b) Verify that $[T(\vec{v})]_{\beta} = [T]_{\beta} [\vec{v}]_{\beta}$

for $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

⑥ Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operator defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2y \\ -y \end{pmatrix}$$

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ and $\beta' = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right]$ be ordered bases for \mathbb{R}^2 ,

(a) Calculate $[T]_{\beta}$

(b) Calculate the change of basis matrix $Q = [I]_{\beta}^{\beta'}$ from β to β' .

(c) Show that $Q[\vec{v}]_{\beta} = [\vec{v}]_{\beta'}$, for $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(d) and $\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$. (d) Show that $[T]_{\beta} = Q^{-1}[T]_{\beta'}Q$

⑦ Calculate the change of basis matrix from β to β' where

$$(a) V = \mathbb{R}^3, \beta = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$\text{and } \beta' = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]$$

$$(b) V = \mathbb{R}^2, \beta = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

$$\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right].$$

(a) ⑧ ~~Using~~ Using your answer $Q = [I]_{\beta}^{\beta'}$ from 7(a) Show that $Q[\vec{v}]_{\beta} = [\vec{v}]_{\beta'}$ where $\vec{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

(b) Using your answer $Q = [I]_{\beta}^{\beta'}$ from 7(b) Show that $Q[\vec{v}]_{\beta} = [\vec{v}]_{\beta'}$ where $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.